If $\omega_1^2 = \omega_2^2 = \omega_3^2$, then the solution exists (see Note 1). Let us consider the case when $\omega_1^2 \neq \omega_2^2$ (similarly we can consider the cases $\omega_1^2 \neq \omega_3^2$ and $\omega_2^2 \neq \omega_3^2$). Equations (3.9) and (3.13) representing a system of two linear equations for three unknown diagonal elements of the matrix B have compatible solutions if $\omega_1^2 \neq \omega_2^2$, since in this case the rank of the matrix of the coefficients accompanying the unknowns, is two. Equations (3.10) and (3.12) representing a system of two linear equations for three unknown products $b_{12}b_{21}$, $b_{13}b_{31}$ and $b_{23}b_{32}$ have compatible solutions for any diagonal elements of the matrix B and any value of the product $b_{12}b_{21}$. Let us set $b_{12} = 0$. Then Eq. (3.11) has a solution relative to b_{21} since one can always make $b_{13}b_{32} \neq 0$. This proves that a solution exists, therefore the upper limit of the degree of stability which is equal to $(\omega_1\omega_2\omega_3)^{1/3}$, can be attained.

In the case of Problem 2, the attainability of the upper limit can be proved in the same manner.

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DYNAMICS OF UNWINDING A FILAMENT

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The motion of a heavy flexible filament being unwound from a rotor is investigated. The aerodynamic drag is taken into account. The possibility is shown of realizing a steady-state process and its investigation is given.

Rapidly-rotating rotors are often fabricated by means of multilayer filament windings [1]. When one of the peripheral turns is ruptured, the effect of aerodynamic drag can prevent complete unwinding of the filament. It is of interest to investigate the possibility of a stationary rotational process for an incompletely unwound filament in the case of a constant angular rotor velocity and the effect of aerodynamic drag, and also to determine the shape and tension of the free part of the filament (not lying on the rotor), the limit radius of the unwinding and the force of interaction between this part of the filament and the rotor.

We consider the stationary process to be possible, and let us find the shape and tension of the free part of the filament. We assume that the filament performs planar motion by rotating at a constant angular velocity equal to the rotor velocity. We neglect the transverse dimensions of the filament. We consider the filament flexible, and the radius of the rotor to equal unity.

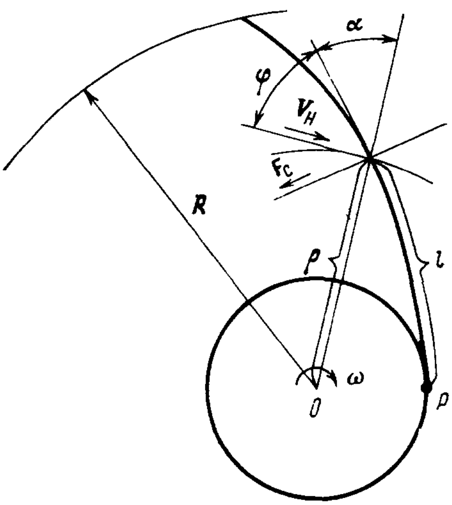

Fig. 1

Let us consider the aerodynamic drag, referred to unit length, which acts on a filament element, to be directed along the normal to this element and to equal

$$F_c = A \mid \mathbf{V}_н - \mathbf{V}_0 \mid^2 \sin^2 \varphi, \quad A = \text{const} \tag{1}$$

Here $\mathbf{V}_н$ is the filament velocity, $\mathbf{V}_0$ is the stream velocity, and φ is the angle between the tangent to the element and the direction of the relative velocity of the filament. This model of the aerodynamic drag is used extensively in investigations of the shape of a fixed filament in an air stream (see [2], p. 105).

Considering the equilibrium of the filament element (Fig. 1) and pojecting the inertial and external forces on the normal and the tangent to the filament (taking into account that F_c is directed along the normal to the filament), we have

$$T_l' = -\gamma\, \omega^2 \rho \cos \alpha \tag{2}$$

$$TK = \gamma\, \omega^2 \rho \sin \alpha - F_c \tag{3}$$

Here ω is the angular rotor velocity, γ is the filament density, i.e. the mass per unit length of filament, ρ is the radius-vector of the filament element with origin at the center of rotation, α is the angle between the tangent to the element and ρ, l is the arc-length measured from the point P, K is the filament curvature, and T is the filament tension.

Let R be the magnitude of the greatest filament radius-vector which we shall call the grazing radius. Then, noting that $\cos \alpha = \rho_l'$, and integrating (2) taking account of the condition $T = 0$ at $\rho = R$, we have

$$T(\rho) = {}^1/_2\, \gamma\omega^2 (R^2 - \rho^2) \tag{4}$$

We note that the tension of an outer turn of the filament lying on the rotor should be greater than $\gamma\omega^2$, otherwise this turn could not stay on the rotor. Now, taking into account that the filament tension has no jump at the point P, i.e. at the point of tangency of the free section with the rotor, and also recalling that the radius of the rotor is equal to unity, we obtain ${}^1/_2\, \gamma\omega^2 (R^2 - 1) \geqslant \gamma\omega^2$ from (4). We hence find that $R \geqslant \sqrt{3}$, in the case of a stationary process, no matter what the aerodynamic drag.

Let us find F_c as a function of ρ and α. We consider the rotor velocity to be sufficiently high so that the boundary layer thickness on the rotor could be neglected and the air surrounding the rotor can be assumed immobile. Now, taking into account that $V_H = \omega\rho$, we write (1) in the form

$$F_c = A\omega^2\rho^2 \sin^2\varphi = A\omega^2\rho^2\cos^2\alpha \tag{5}$$

Moreover, noting that $K = \rho^{-1}(\rho\sin\alpha)_\rho'$ (see [3, 4]), we obtain from (2)

$$\frac{R^2-\rho^2}{2\rho}(\rho\sin\alpha)_\rho' = \rho\sin\alpha - a\rho^2\cos^2\alpha, \quad a = \frac{A}{\gamma}$$

We obtain the Riccati equation from this latter by substituting $y = \rho\sin\alpha$

$$y_\rho' = 2\rho\,\frac{y+ay^2-a\rho^2}{R^2-\rho^2} \tag{6}$$

Here there is an unknown constant R (the grazing radius) in the right side, which can be determined after integrating (6) with the following boundary conditions

$$y = 1 \quad \text{for} \quad \rho = 1 \tag{7}$$

$$y + ay^2 - a\rho^2 = 0 \quad \text{for} \quad \rho = R \tag{8}$$

Condition (7) means that the free section of the filament has no break at the point of contact with the rotor, and the filament is tangent to the rotor. Condition (8) means that the derivative y_ρ' is bounded for $\rho = R$, i.e. the curvature is finite at the point $\rho = R$.

Introducing the new variables t and $u(t)$, connected to ρ and y by the relationships

$$t = 2a\sqrt{R^2-\rho^2}, \quad \frac{t}{2a}u_t' = uy \tag{9}$$

we have from (6)

$$tu_{tt}'' + 3\,tu_t' + (t - 4\,a^2R^2)\,u = 0 \tag{10}$$

Let us still consider the quantity $\sqrt{1+4\,a^2R^2}$ not to be an intger. Then integrating, we have

$$u = \frac{1}{t}\left[c_1J_\nu + c_2J_{-\nu}\right], \quad \nu = \sqrt{1+4a^2R^2} \tag{11}$$

Here c_1 and c_2 are constants of integration, $J_\nu = J_\nu(t)$, $J_{-\nu} = J_{-\nu}(t)$ are Bessel functions of order ν and $-\nu$.

Taking into account the recurrence relationships for the Bessel functions and returning to y, we obtain

$$2ay = -1 + \frac{\nu(-c_1J_\nu + c_2J_{-\nu}) + t\,(c_1J_{\nu-1} + c_2J_{-\nu-1})}{c_1J_\nu + c_2J_{-\nu}} \tag{12}$$

Condition (8) can be rewritten as follows:

$$2ay\,|_{\rho=R} = -1 \pm \sqrt{1+4a^2R^2}$$

Let us consider $\sin\alpha > 0$ (this inequality is conserved particularly when the curve has no inflection). Then the function y is positive everywhere and, taking account of (11), we can write the last condition thus:

$$2ay\,|_{t=0} = \nu - 1 \tag{13}$$

Only $\nu > 1$ has physical meaning, and besides, we still consider noninteger ν, hence, by dividing numerator and denominator of the last member in (12) by $J_{-\nu}$ we obtain

$$\lim_{t\to 0} y(t) = \frac{1}{2a}(-\nu - 1) \quad \text{for} \quad c_2 \neq 0$$

Therefore, condition (13) is not satisfied for $c_2 \neq 0$. Consequently, $c_2 = 0$. After substituting $c_2 = 0$ into (12), c_1 is also canceled. As a result we have

$$y = \frac{1}{2a}\left[-1 - \nu + t\,\frac{J_{\nu-1}(t)}{J_\nu(t)}\right] \tag{14}$$

Here y, t and ν are expressed in terms of the initial quantities ρ and α as follows:

$$t = 2a\sqrt{R^2-\rho^2}, \quad y = \rho\sin\alpha, \quad \nu = \sqrt{1+4a^2R^2} \tag{15}$$

Taking into account that $\lim_{t\to 0} tJ_{\nu-1}/J_\nu = 2\nu$, it can be noted that this solution satisfies condition (13).

In the case of integer ν we find that (14) is valid also for integer ν by representing the solution of (10) as the sum of Bessel and Neumann functions and performing analogous computations.

The quantity ν remains unknown in (14) and should be found from the condition (7). As is seen from (9), the value $\rho = 1$ corresponds to the value $t = t_0 = 2a\sqrt{R^2-1}$.

Consequently, ν is found from the following system of transcendental equations:

$$2a + 1 + \nu = t_0 J_{\nu-1}(t_0)/J_\nu(t_0), \quad t_0 = 2a\sqrt{R^2-1}, \quad \nu = \sqrt{1+4aR^2} \tag{16}$$

Therefore, we hence find ν for a given value a, where we incidentally find the magnitude of the grazing radius R. According to (14) the tension of each point of the filament will consequently be known, as will the tension at the point of contact with the rotor also, i.e. the force and moment with which the free section of the filament act on the rotor.

For values of ν equal to half an odd number, the function J_ν and therefore y also are expressed in terms of elementary functions. For $\nu = 1.5$ we have

$$J_{\nu-1}(t)/J_\nu(t) = t\sin t/(\sin t - t\cos t)$$

Using the recurrence relationship

$$J_{\nu-1}(t)/J_\nu(t) = t\,[2\nu - t\,J_{\nu-2}(t)/J_{\nu-1}(t)]^{-1}$$

we find y in (14) for each successive ν.

The first of equations (16) becomes

$$2a + 2.5 = \frac{t_0^2\sin t_0}{\sin t_0 - t_0\cos t_0} \quad \text{for } \nu = 1.5$$

$$2a + 3.5 = \frac{t_0^2(\sin t_0 - t_0\cos t_0)}{(3-t_0^2)\sin t_0 - 3t_0\cos t_0} \quad \text{for } \nu = 2.5$$

etc. Consequently, we obtain from (16) that $a = 0.12$, $R^2 = 21$ for $\nu = 1.5$, correspond to $a = 0.38$, $R^2 = 9.1$ for $\nu = 2.5$, $a = 0.75$, $R^2 = 5$ for $\nu = 3.5$, etc. Therefore, the grazing radius decreases as the aerodynamic drag grows.

The tension at the point of contact P is computed by means of (4). Substituting $\rho = 1$ and the values of R found above into (4), we obtain that the quantity $T(1)/(\gamma\omega^2)$ equals 10, 4.1, 2.5, etc. for $a = 0.12, 0.38, 0.75$, etc. As we see $T(1)$, and therefore, the moment acting from the filament on the rotor also diminish as the aerodynamic drag grows.

Let us examine the case of small a, which corresponds to a large specific mass of filament or small aerodynamic drag.

The quantity $t_0 J_{\nu-1}/J_\nu$ in (16) can be represented as follows:

$$t_0\,\frac{J_{\nu-1}(t_0)}{J_\nu(t_0)} = 2\nu\,\frac{1-(t_0/2)^2\nu^{-1} + S(t_0)}{1-(t_0/2)^2(\nu+1)^{-1} + s(t_0)}, \quad S = O(t_0^4), \quad s = O(t_0^4) \tag{17}$$

Taking account of this expansion in an investigation of the system (16) as $t_0 \to 0$, we find

$$t_0 \to 0, \quad \nu \to 1, \quad aR^2 \to 2 \quad \text{for } a \to 0 \tag{18}$$

Therefore $R \to \infty$ as $a \to 0$, i.e. as should have been expected, there can be no stationary process for an incompletely unwound filament for $a = 0$ (which corresponds to filament rotation in a vacuum). We find from (4) that $Ta \to \gamma\omega^2$ as $a \to 0$, $\rho \neq R$, i.e. the tension at each fixed radius increases without limit as a diminishes.

Since $t_0 \to 0$ as $a \to 0$, then we have $t_0 / 2 < 1$ for sufficiently small a. Consequently, according to the Leibnitz rule for an alternating-sign decreasing series, for sufficiently small a, we have according to (17)

$$0 < s < \frac{(t_0 / 2)^4}{2(\nu + 2)(\nu + 1)}, \qquad S > s$$

We now obtain from (17)

$$t_0 \frac{J_{\nu-1}(t_0)}{J_\nu(t_0)} > 2\nu \frac{1 - (t_0 / 2)^2 \nu^{-1}}{1 - (t_0 / 2)^2 (\nu + 1)^{-1}}$$

Substituting this inequality into (16), we have (taking into account that $\nu > 1$)

$$\nu < 2 + a - \sqrt{a^2 + 1 - 6a}$$

Since $\nu = \sqrt{1 + 4a^2R^2}$, then

$$R^2 < \frac{1}{2a^2}[2 - a + a^2 - (2 + a)\sqrt{a^2 + 1 - 6a}]$$

For example, let $a \leqslant 0.12$. Then

$$R^2 < 2 / a + 9.5 \tag{19}$$

We have $R^2 < 25$ from this inequality for $a = 0.12$, while the exact value of R^2 found above is 21 in this case. Thus, the grazing radius R which grows without limit as a decreases, remains less than the quantity $\sqrt{2 / a + 9.5}$ in conformity with (18).

From (4) and (19) we obtain the following inequality for tension at the point P (for $\rho = 1$):

$$T|_{\rho=1} < \gamma\omega^2 (1 / a + 4.25)$$

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